CentER

Discussion Paper

No. 2007–54

EGALITARIANISM IN MULTI-CHOICE GAMES

By R. Branzei, N. Llorca, J. Sánchez-Soriano, S. Tijs

July 2007

ISSN 0924-7815

TILBURG UNIVERSITY

Egalitarianism in multi-choice games

R. Branzei

Faculty of Computer Science,
"Alexandru Ioan Cuza" University, Iaşi, Romania

N. Llorca,* J. Sánchez-Soriano* †

CIO and Department of Statistics, Mathematics and Computer Science
University Miguel Hernández of Elche, Spain

S. Tijs*

CentER and Department of Econometrics
and Operations Research,
Tilburg University, The Netherlands

Abstract

In this paper we introduce the equal division core for arbitrary multi-choice games and the constrained egalitarian solution for convex multi-choice games, using a multi-choice version of the Dutta-Ray algorithm for traditional convex games. These egalitarian solutions for multi-choice games have similar properties as their counterparts for traditional cooperative games. On the class of convex multi-choice games, we axiomatically characterize the constrained egalitarian solution.

JEL Classification: C71.

Keywords: Multi-choice games, Convex games, Equal division core, Constrained egalitarian solution.

---

*These authors acknowledge the financial support from the Governement of Spain and FEDER fund under project MTM 2005-09184-C02-02 and from Generalitat Valenciana under project ACOMP07/136.

†Corresponding author, e-mail: joaquin@umh.es

1 Introduction

Multi-choice cooperative games introduced by Hsiao and Raghavan (1993a,b) and van den Nouweland et al. (1995) are natural extensions of traditional cooperative games. Whereas in a traditional cooperative game each player may have only two options concerning cooperation, being either active or inactive, in a multi-choice context each player may have additional participation opportunities in a finite set of activity levels. Recall that a traditional cooperative game is a pair $\langle N, v\rangle$, where N is the set of players, usually of the form $\{1, 2, ..., n\}$ and $v : 2^N \to \mathbb{R}$ with $v(\emptyset) = 0$ is the characteristic function. Each element $S \in 2^N$ is called a *coalition* and $v(S)$ measures the reward reachable by S on its own. Often, we identify a game $\langle N, v\rangle$ with its characteristic function v. We denote here by G^N the set of characteristic functions with player set N. A basic issue in traditional game theory is how to distribute the worth $v(N)$ of the grand coalition N among the players, when the grand coalition N forms. We refer to such a distribution as an efficient payoff vector. When such a payoff vector is also individual rational, i.e. it guarantees to each player $i \in N$ at least the amount $v(\{i\})$, it is called an imputation. The set of imputations of a game $v \in G^N$ is usually denoted by $I(v)$, where

$$I(v) = \left\{ x \in \mathbb{R}^n \mid \sum_{i=1}^{n} x_i = v(N) \text{ and } x_i \geq v(\{i\}) \text{ for each } i \in N \right\}.$$

The concept of egalitarianism has generated several so-called egalitarian solutions. We mention here for arbitrary cooperative games: the equal division core (Selten, 1972), the constrained egalitarian solution (Dutta and Ray, 1989), the strong-constrained egalitarian allocations (Dutta and Ray, 1991), the egalitarian set, the preegalitarian set and the stable egalitarian set (Arin and Inarra, 2002), and the equal split-off set (Branzei, Dimitrov and Tijs, 2006). A central solution concept in cooperative game theory is the core (Gillies, 1953). For any game $v \in G^N$ the core $C(v)$ of v is defined by

$$C(v) = \left\{ x \in I(v) \mid \sum_{i \in S} x_i \geq v(S) \text{ for each } S \in 2^N \right\}.$$

The core of a cooperative game v is included in the equal division core $EDC(v)$ of v which is the set

$$EDC(v) = \left\{ x \in I(v) \mid \nexists\, S \in 2^N \setminus \{\emptyset\} \text{ s.t. } \frac{v(S)}{|S|} > x_i \text{ for all } i \in S \right\},$$

consisting of efficient payoff vectors which cannot be improved upon by the equal division allocation of any subcoalition. Note that in the definition of $EDC(v)$, the *average worth* $a(S,v) := v(S)/|S|$ of coalition S with respect to v, called also the *per capita value* of S with respect to v, plays a role. Axiomatic characterizations of the equal division core on two classes of cooperative games can be found in Bhattacharya (2004). The well-known Bondareva-Shapley theorem (Bondareva, 1963; Shapley, 1967) states that a game is balanced iff its core is non-empty. For balanced games, interesting egalitarian solution concepts are: the Lorenz stable set, the leximin stable allocation, the egalitarian core (Arin and Inarra, 2001) and the Lorenz solution (Hougaard et al., 2001). In most of the above mentioned egalitarian solutions, an egalitarian criterion, namely the Lorenz criterion, plays a central role. The Lorenz criterion is based on the so-called Lorenz domination.

Let $x, y \in \mathbb{R}^n$ with $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i = I \in \mathbb{R}$. Denote by $\hat{x} = (\hat{x}_1, ..., \hat{x}_n)$ and $\hat{y} = (\hat{y}_1, ..., \hat{y}_n)$ the vectors obtained from x and y by rearranging their coordinates in non-decreasing order, that is $\hat{x}_1 \leq \hat{x}_2 \leq ... \leq \hat{x}_n$ and $\hat{y}_1 \leq \hat{y}_2 \leq ... \leq \hat{y}_n$. We say that x *Lorenz dominates* y, and denote this by $x \succ_L y$, iff $\sum_{i=1}^p \hat{x}_i \geq \sum_{i=1}^p \hat{y}_i$ for all $p \in \{1, ..., n-1\}$, with at least one strict inequality.

The Lorenz domination yields a measure of inequality; for this and other measures of inequality the reader is referred to Atkinson (1970), Dasgupta, Sen and Starret (1973), Fields and Fei (1978) and Sen (1973, 1997).

In case the characteristic function of a game v is supermodular, i.e. $v(S \cup T) + v(S \cap T) \geq v(S) + v(T)$ for all $S, T \in 2^N$, the game is called convex (Shapley, 1971). Convex games are balanced games; moreover, they have large cores (Sharkey, 1982). For convex games the constrained egalitarian solution exists, belongs to the core and Lorenz dominates each other core allocation. For further use we denote the class of convex games with player set N by CG^N. Furthermore, all the above mentioned egalitarian solutions, except the equal division core, coincide with the constrained egalitarian solution on the class of convex games. An algorithm, which we call henceforth the Dutta-Ray algorithm (Dutta and Ray, 1989), and a formula (Hokari, 2000) for computing the constrained egalitarian solution of a game $v \in CG^N$ are available. Both proved to be useful tools to show that the constrained egalitarian solution for such games possesses a population monotonicity property (see Lemma 5.5 and Remark 5.1 in Dutta (1990) and Section 3 in Hokari (2000)) which tells us that when new players arrive, the original players are weakly better off. We refer the reader to Thomson (1995) for a survey of results concerning this property in game theory and several models of economics. The Dutta-Ray algorithm for convex games is suitable whenever the players believe in equal share cooperation and agree upon successive weak split-off of largest groups with highest equal share, as soon as each player in such a group receives the maximal equal share at stake. We notice that after each round of equal share distribution, the players remained in the allocation process play a related (convex) marginal game by applying the equal share principle in a consistent way. Axiomatic characterizations of the constrained egalitarian solution on the class CG^N can be found in Dutta (1990) and Klijn, Slikker and Zarzuelo (2000).

The main purpose of this paper is to introduce and study the equal division core for arbitrary multi-choice games (cf. van den Nouweland et al., 1995) and the constrained egalitarian solution for convex multi-choice games, and to cope with the equal split-off set for arbitrary multi-choice games in an informal way.

The outline of the paper is as follows. Section 2 provides basic definitions, notations and results on (convex) multi-choice games. In Section 3, the equal division core for multi-choice games is introduced and it is shown that it is a refinement of the precore (Grabisch and Xie, 2007), which is a straightforward generalization of the core for traditional games. Section 4 is devoted to the constrained egalitarian solution for convex multi-choice games, which is obtained by adjusting for convex multi-choice games the Dutta-Ray algorithm for games $v \in CG^N$. We prove that the constrained egalitarian allocation of a convex multi-choice game belongs to the precore of the game and Lorenz dominates each other precore allocation. Moreover, it turns out that the constrained egalitarian solution of a convex multi-choice game belongs to the equal division core of the game. This property, which we call the equal division stability, together with the efficiency property and a max-consistency property are used to axiomatically characterize the constrained egalitarian solution on the class of convex multi-choice games. In Section 5 we briefly discuss the extension of the equal split-off set for arbitrary traditional games (cf. Branzei, Dimitrov and Tijs, 2006) to arbitrary multi-choice games and its properties.

2 Preliminaries on multi-choice cooperative games

Let N be a set of players that consider cooperation in a multi-choice environment, i.e. each player $i \in N$ has a finite number of feasible participation levels whose set we denote by $M_i = \{0, 1, ..., m_i\}$. We consider their product $\mathcal{M}^N = \prod_{i \in N} M_i$. Each element $s = (s_1, s_2, ..., s_n) \in \mathcal{M}^N$ specifies a participation profile for players and is referred to as a multi-choice coalition. Then, $m = (m_1, m_2, ..., m_n)$ is the players' maximal participation level profile that plays the role of the "grand coalition", whereas $0 = (0, 0, ..., 0)$ plays the role of the "empty coalition". We also use the notation $M_i^+ = M_i \setminus \{0\}$ and $\mathcal{M}_+^N = \mathcal{M}^N \setminus \{0\}$. A cooperative multi-choice game is a triple $\langle N, m, v\rangle$, where $v : \mathcal{M}^N \to \mathbb{R}$ is the characteristic function with $v(0) = 0$ that specifies the players' potential worth, $v(s)$, when they join their efforts at any activity level profile $s = (s_1, ..., s_n)$. We note that games $v \in G^N$ can be seen as special multi-choice games $\langle N, m, v\rangle$ where $m = (1, 1, ..., 1)$. For $s \in \mathcal{M}^N$ we denote by (s_{-i}, k) the participation profile where all players except player i play at levels defined by s while player i plays at level k in M_i. A useful particular case is $(0_{-i}, k)$, when only player i is active. We define the carrier of s by $\mathrm{car}(s) = \{i \in N \mid s_i > 0\}$. For $s, t \in \mathcal{M}^N$ we use the notation $s \le t$ iff $s_i \le t_i$ for each $i \in N$ and define $s \wedge t = (\min(s_1, t_1), ..., \min(s_n, t_n))$ and $s \vee t = (\max(s_1, t_1), ..., \max(s_n, t_n))$. We denote the set of all multi-choice games with player set N and maximal participation profile m by $MC^{N,m}$. Often, we identify a multi-choice game $\langle N, m, v\rangle$ with its characteristic function v. A game $v \in MC^{N,m}$ is called *convex* if

$$v(s \wedge t) + v(s \vee t) \ge v(s) + v(t) \text{ for all } s, t \in \mathcal{M}^N. \tag{2.1}$$

Relation (2.1) is equivalent to

$$v(s + t) - v(s) \ge v(\bar{s} + t) - v(\bar{s}) \tag{2.2}$$

for all $s, \bar{s}, t \in \mathcal{M}^N$ satisfying $\bar{s} \le s$, $\bar{s}_i = s_i$ for all $i \in \mathrm{car}(t)$ and $s + t \in \mathcal{M}^N$. In the sequel, we denote the class of convex multi-choice games with player set N and maximal participation profile m by $CMC^{N,m}$.

Let $v \in MC^{N,m}$. We define $M := \{(i, j) \mid i \in N,\ j \in M_i\}$ and $M^+ := \{(i, j) \mid i \in N,\ j \in M_i^+\}$. A (level) payoff vector for the game v is a function $x : M \to \mathbb{R}$, where for each $i \in N$ and each $j \in M_i^+$, x_{ij} denotes the increase in payoff for player i corresponding to a change of his activity level from

$j-1$ to j, and $x_{i0}=0$ for all $i \in N$. One can represent a payoff vector for a game v as a $\sum_{i\in N} m_i$-dimensional vector whose coordinates are numbered by the corresponding elements of M^+, where the first m_1 coordinates represent payoffs for successive level increases of player 1, the next m_2 coordinates are payoffs for successive level increases of player 2, and so on. A level payoff vector $x : M \to \mathbb{R}$ is called

- *efficient* if $X(m) = \sum_{i\in N} \sum_{j=1}^{m_i} x_{ij} = v(m)$;
- *level-increase rational* if, for all $i \in N$ and $j \in M_i^+$, x_{ij} is at least the increase in worth that player i can obtain on his own (i.e. working alone) when he changes his activity level from $j-1$ to j, that is $x_{ij} \geq v(je^i) - v((j-1)e^i)$, or, equivalently, $x_{ij} \geq v(0_{-i}, j) - v(0_{-i}, j-1)$.

A payoff vector which is both efficient and level-increase rational is called an *imputation*. We denote by $I(v)$ the set of imputations of $v \in MC^{N,m}$.

3 The equal division core for multi-choice games

The core $C(v)$ of a game $v \in MC^{N,m}$ consists of all $x \in I(v)$ that satisfy $X(s) \geq v(s)$ for all $s \in \mathcal{M}^N$, i.e.

$$C(v) = \{x \in I(v) \mid X(s) \geq v(s) \text{ for all } s \in \mathcal{M}^N\}.$$

A game whose core is nonempty is called a *balanced game*. Clearly, for each $v \in MC^{N,m}$, the set $C(v)$ is a convex set.

In this paper, a notion related to the core, which is called the precore (cf. Grabisch and Xie (2007)) and is a direct generalization of the core for traditional cooperative games, will play a central role. The *precore* $\mathcal{PC}(v)$ of $v \in MC^{N,m}$ is defined by

$$\mathcal{PC}(v) = \{x : M \to \mathbb{R} \mid X(m) = v(m) \text{ and } X(s) \geq v(s) \text{ for all } s \in \mathcal{M}^N\}.$$

Clearly, the precore $\mathcal{PC}(v)$ of v is a convex polyhedron with infinite directions which includes the (unbounded) set $C(v)$. (Note that precore allocations are not necessarily imputations). A game $v \in MC^{N,m}$ is called a *pre-balanced game* iff it has a non-empty precore.

In this paper, the marginal game of a multi-choice game with respect to a multi-choice coalition, introduced by Branzei, Tijs and Zarzuelo (2007), will play an important role, too. Let $v \in MC^{N,m}$ and let $u \in \mathcal{M}_+^N$. We denote by $\mathcal{M}_u^N$ the subset of $\mathcal{M}^N$ consisting of multi-choice coalitions $s \leq u$. The *marginal game* of v based on u (or, shortly, the u-marginal game of v), is the multi-choice game $\langle N, m-u, v^{-u}\rangle$, where $v^{-u}(s) := v(s+u) - v(u)$ for each $s \in \mathcal{M}_{m-u}^N$. For each $v \in CMC^{N,m}$ and each $u \in \mathcal{M}_+^N$, it holds true that $v^{-u} \in CMC^{N,m-u}$ (see Lemma 3.1 in Branzei, Tijs and Zarzuelo (2007)).

For $v \in MC^{N,m}$ and $t \in \mathcal{M}_+^N$ we also need the notation $M_i^t = \{1, ..., t_i\}$ for each $i \in N$ and $\mathcal{M}_t^N = \prod_{i \in N} M_i^t$.

Let $v \in MC^{N,m}$ and let $s \in \mathcal{M}^N$. Let $\|s\|_1 = \sum_{i=1}^n s_i$ be the cumulate number of levels of players according to the participation profile s. Given $v \in MC^{N,m}$ and $s \in \mathcal{M}_+^N$, we denote by $\alpha(s,v)$ the (per levels) average worth of s with respect to v, i.e.

$$\alpha(s,v) := \frac{v(s)}{\|s\|_1} \cdot$$

Note that $\alpha(s,v)$ can be interpreted as a per one-unit level increase value of coalition s. Given a cooperative multi-choice game $v \in MC^{N,m}$, we define the *equal division core* $EDC(v)$ of v as the set $\{x : M \to \mathbb{R} \mid X(m) = v(m);$ $\nexists s \in \mathcal{M}_+^N$ s.t. $\alpha(s,v) > x_{ij}$ for all $i \in car(s),\ j \in M_i^+\}$. So, $x \in EDC(v)$ can be seen as a distribution of the value of the grand coalition m, where for each multi-choice coalition s, there exists a player i with a positive participation level in s and an activity level $j \in M_i^+$ such that the payoff x_{ij} is at least as good as the equal division share $\alpha(s,v)$ of $v(s)$.

The relation between the equal division core of a game $v \in G^N$ and the corresponding core can be extended for multi-choice games with the precore of such games in the role of the core for games in $v \in G^N$.

Theorem 3.1 *Let $v \in MC^{N,m}$. Then $\mathcal{PC}(v) \subset EDC(v)$.*

Proof. Let $x \in \mathcal{PC}(v)$ and let us suppose that $x \notin EDC(v)$. Then there exists $s \in \mathcal{M}_+^N$ such that $\alpha(s,v) > x_{ij}$ for all $i \in car(s)$ and $j \in \{1, ..., s_i\}$. Then

$$X_{is_i} = \sum_{j=1}^{s_i} x_{ij} < s_i \alpha(s,v), \text{ implying that}$$
$$X(s) = \sum_{i \in N} X_{is_i} < \sum_{i \in N} s_i \alpha(s,v) = \alpha(s,v) \cdot \sum_{i \in N} s_i = v(s).$$

So, $x \notin \mathcal{PC}(v)$. ∎

We notice that the inclusion relation in Theorem 3.1 may be strict, as in the case of traditional cooperative games (which are special multi-choice games).

Corollary 3.1 *For each $v \in MC^{N,m}$, $C(v) \subset \mathcal{PC}(v) \subset EDC(v)$.*

4 The constrained egalitarian solution for convex multi-choice games

Now, we introduce the multi-choice counterpart of the constrained egalitarian solution of a game $v \in CG^N$ by using an adjusted version of the Dutta-Ray algorithm (Dutta and Ray, 1989).

To formulate the Dutta-Ray algorithm in a multi-choice setting we need to prove that for each convex multi-choice game there exists a unique multi-choice coalition with the largest accumulated number of one-unit level increases of players among all coalitions with the highest (per one-unit level increase) average worth.

Lemma 4.1 *Let $v \in CMC^{N,m}$. Then, the set*

$$A(v) := \left\{ t \in \mathcal{M}_+^N \mid \alpha(t,v) = \max_{s \in \mathcal{M}_+^N} \alpha(s,v) \right\}$$

is closed with respect to the join operator $\vee$.

Proof. Let $\bar{\alpha} = \max_{s \in \mathcal{M}_+^N} \alpha(s,v)$ and take $t^1, t^2 \in A(v)$. We have to prove that $t^1 \vee t^2 \in A(v)$, that is $\alpha(t^1 \vee t^2, v) = \bar{\alpha}$. Since $v(t^1) = \bar{\alpha}\|t^1\|_1$ and $v(t^2) = \bar{\alpha}\|t^2\|_1$, we obtain

$$\begin{aligned} &\bar{\alpha}\|t^1\|_1 + \bar{\alpha}\|t^2\|_1 = v(t^1) + v(t^2) \leq v(t^1 \vee t^2) + v(t^1 \wedge t^2) \\ &\leq \bar{\alpha}\|t^1 \vee t^2\|_1 + \bar{\alpha}\|t^1 \wedge t^2\|_1 = \bar{\alpha}\|t^1\|_1 + \bar{\alpha}\|t^2\|_1, \end{aligned}$$

where the first inequality follows from the convexity of v, and the second inequality follows from the definition of $\bar{\alpha}$ and the fact that $v(0) = 0$ (in case $t^1 \wedge t^2 = 0$). This implies that $v(t^1 \vee t^2) = \bar{\alpha}\|t^1 \vee t^2\|_1$. Hence, $t^1 \vee t^2 \in A(v)$, in case $t^1 \wedge t^2 \in A(v)$ as well as in case $\|t^1 \wedge t^2\|_1 = 0$. ∎

We can conclude from the proof of Lemma 4.1 that for any $t^1, t^2 \in A(v)$ not only $t^1 \vee t^2 \in A(v)$ holds true, but also $t^1 \wedge t^2 \in A(v)$ if $t^1 \wedge t^2 \neq 0$. Further, $A(v)$ is closed with respect to finite "unions", where $t^1 \vee t^2$ is seen as the "union" of t^1 and t^2. Thus, Proposition 4.1 holds true.

Proposition 4.1 *Let $v \in CMC^{N,m}$. Then, there exists a unique element in $\arg\max_{s \in \mathcal{M}_+^N} \alpha(s,v)$ with the maximal number of cumulate one-unit level increases.*

Proof. The set $A(v) \cup \{0\}$ has a lattice structure and $\bigvee_{t \in A(v)} t$ is the largest element in $A(v)$. ∎

Now, we introduce, in a similar way to that of Dutta and Ray (1989), an egalitarian rule on the class of convex multi-choice games. In view of Lemma 3.1 (in Branzei, Dimitrov and Tijs, 2007), Lemma 4.1 and Proposition 4.1, it is easy to adjust the Dutta-Ray algorithm for convex multi-choice games. In Step 1, one puts $m^1 := m$, $v_1 := v$, and considers the unique element in $\arg\max_{s \in \mathcal{M}_{m^1}^N \setminus \{0\}} \alpha(s, v_1)$ with the maximal cumulate number of one-unit level increases, say s^1. Define $d_{ij} := \alpha(s^1, v_1)$ for each $i \in car(s^1)$ and $j \in M_i^{s^1}$. If $s^1 = m$, then we stop. Otherwise, in Step 2, we consider the convex multi-choice game $\langle N, m^2, v_2 \rangle$, where $m^2 := m^1 - s^1$ and for each $s \in \mathcal{M}_{m^2}^N$, $v_2(s) := v_1(s + s^1) - v_1(s^1)$. Once again, by using Proposition 4.1, we can take the largest element s^2 in $\arg\max_{s \in \mathcal{M}_{m^2}^N \setminus \{0\}} \alpha(s, v_2)$ and define $d_{ij} := \alpha(s^2, v_2)$ for all $i \in car(s^2)$ and $j \in \{s_i^1 + 1, ..., s_i^1 + s_i^2\}$. If $s^1 + s^2 = m$ we stop; otherwise we continue by considering the multi-choice game $\langle N, m^3, v_3 \rangle$, etc.

Step p: Suppose that $s^1, s^2, ..., s^{p-1}$ have been defined recursively and $s^1 + s^2 + ... + s^{p-1} \neq m$. We define a new multi-choice game with player set N and maximal participation profile $m^p := m - \sum_{i=1}^{p-1} m^i$. For each multi-choice coalition $s \in \mathcal{M}_{m^p}^N$, we define $v_p(s) := v_{p-1}(s + s^{p-1}) - v_{p-1}(s^{p-1})$. The game $\langle N, m^p, v_p \rangle$ is convex. We denote by s^p the (unique) largest element in $\arg\max_{s \in \mathcal{M}_{m^p}^N \setminus \{0\}} \alpha(s, v_p)$ and define $d_{ij} := \alpha(s^p, v_p)$ for all $i \in car(s^p)$ and $j \in \left\{ \sum_{k=1}^{p-1} s_i^k + 1, ..., \sum_{k=1}^{p} s_i^k \right\}$.

In $P \leq |M^+|$ steps the algorithm will end, and the constructed allocation $(d_{ij})_{(i,j) \in M^+}$ is called the (Dutta) constrained egalitarian solution $d(v)$ of the convex multi-choice game v.

Remark 4.1 Note that the above described Dutta-Ray algorithm determines in P steps for each $v \in CMC^{N,m}$ a sequence of (per one-unit level increase) average values $\alpha_1, \alpha_2, ..., \alpha_p$ with $\alpha_p := \alpha(s^p, v_p)$ for each $p \in \{1, ..., P\}$, and a sequence of multi-choice coalitions in $\mathcal{M}_+^N$, which we denote by $t^1 := s^1$, $t^2 := s^1 + s^2$,..., $t^p := s^1 + ... + s^p$, ..., $t^P := s^1 + ... + s^P = m$. Thus, a unique path $\left\langle t^0., t^1, ..., t^P \right\rangle$, with $t^0 = 0$ from 0 to m is obtained, to which we can associate a suitable ordered partition $D^1, D^2, ..., D^P$ of M, such that for all $p \in \{1, ..., P\}$, $D^p := \{(i,j) \mid i \in car(t^p - t^{p-1}),\ j \in \{t_i^{p-1} + 1, ..., t_i^p\}$, where for each $(i,j) \in D^p$, $d_{ij} = \alpha_p$, and the coalition $t^p - t^{p-1}$ is the maximal participation profile in the "box" D^p with average worth α_p. Note that each other participation profile in D^p can be expressed as $s \wedge t^p - s \wedge t^{p-1} + t^{p-1}$, where $s \in \mathcal{M}_+^N$. Clearly, the average worth of such a participation profile is weakly smaller than α_p.

The next example illustrates the Dutta-Ray algorithm for convex multi-choice games.

Example 4.1 Consider the game $\langle N, m, v \rangle$ with $N = \{1, 2\}$, $m = (2, 1)$, $v(0,0) = 0$, $v(1,0) = 3$, $v(2,0) = 4$, $v(0,1) = 2$, $v(1,1) = 8$, $v(2,1) = 10$. The game is convex and we apply the Dutta-Ray algorithm. In Step 1, $\alpha_1 = 4$, $t^1 = s^1 = (1,1)$, and we have $d_{11} = d_{21} = 4$. In Step 2, $\alpha_2 = 2$, $t^2 = (1,1) + (1,0)$ and we have $d_{12} = 2$. We obtain $d(v) = (4, 2, 4)$. Note that $\alpha_1 > \alpha_2$. This is true in general, as we show in Proposition 4.2.

Proposition 4.2 *Let $v \in CMC^{N,m}$ and let $\alpha_p = \max_{s \in \mathcal{M}_{m^p}^N \setminus \{0\}} \frac{v_p(s)}{\|s\|_1}$ be the egalitarian distribution share determined in Step p of the Dutta-Ray algorithm. Then $\alpha_p \geq \alpha_{p+1}$ for all $p \in \{1, ..., P-1\}$.*

Proof. By definition of v_p and α_p, and in view of Remark 4.1, we have

$$\frac{v(t^p) - v(t^{p-1})}{\|t^p - t^{p-1}\|_1} \geq \frac{v(t^{p+1}) - v(t^{p-1})}{\|t^p - t^{p-1}\|_1 + \|t^{p+1} - t^p\|_1}.$$

By adding and subtracting $v(t^p)$ in the numerator of the right-hand term, we obtain

$$\frac{v(t^p) - v(t^{p-1})}{\|t^p - t^{p-1}\|_1} \geq \frac{v(t^{p+1}) - v(t^p) + v(t^p) - v(t^{p-1})}{\|t^p - t^{p-1}\|_1 + \|t^{p+1} - t^p\|_1}.$$

This inequality is equivalent to

$$\begin{aligned}&(v(t^p) - v(t^{p-1}))\|t^p - t^{p-1}\|_1 + (v(t^p) - v(t^{p-1}))\|t^{p+1} - t^p\|_1 \\ &\geq (v(t^{p+1}) - v(t^p))\|t^p - t^{p-1}\|_1 + (v(t^p) - v(t^{p-1}))\|t^p - t^{p-1}\|_1,\end{aligned}$$

which is, in turn, equivalent to

$$(v(t^p) - v(t^{p-1}))\|t^{p+1} - t^p\|_1 \geq (v(t^{p+1}) - v(t^p))\|t^p - t^{p-1}\|_1. \qquad \blacksquare$$

Next, we prove that the constrained egalitarian solution for convex multi-choice games has similar properties as the constrained egalitarian solution for traditional convex games; in particular, it belongs to the precore and Lorenz dominates each other precore allocation.

Lemma 4.2 *Let $v \in CMC^{N,m}$. Let P be the number of steps in the Dutta-Ray algorithm for constructing the constrained egalitarian solution $d(v)$ of v, and let $t^1, t^2, ..., t^P$ be the corresponding sequence of multi-choice coalitions in $\mathcal{M}_+^N$. Then, for each $s \in \mathcal{M}_+^N$ and each $p \in \{1, ..., P\}$,*

$$v(s \wedge t^p - s \wedge t^{p-1} + t^{p-1}) - v(t^{p-1}) \geq v(s \wedge t^p) - v(s \wedge t^{p-1}).$$

Proof. First, notice that, for each $i \in N$,

$$\min\{s_i, t_i^{p-1}\} = \min\{\min\{s_i, t_i^p\}, t_i^{p-1}\}$$

because $t_i^p \geq t_i^{p-1}$, implying that $s \wedge t^{p-1} = (s \wedge t^p) \wedge t^{p-1}$.

Second, notice that, for $i \in N$, either $\min\{s_i, t_i^{p-1}\}{=}t_i^{p-1}$ or $\min\{s_i, t_i^{p-1}\}{=}s_i$, and in both situations we have

$$\min\{s_i, t_i^p\} - \min\{s_i, t_i^{p-1}\} + t_i^{p-1} = \max\{\min\{s_i, t_i^p\}, t_i^{p-1}\},$$

implying that

$$(s \wedge t^p) - (s \wedge t^{p-1}) + t^{p-1} = (s \wedge t^p) \vee t^{p-1}.$$

Now, by convexity of v (with $s \wedge t^p$ in the role of s and t^{p-1} in the role of t), we obtain

$$v((s \wedge t^p) \vee t^{p-1}) + v(s \wedge t^{p-1}) \geq v(s \wedge t^p) + v(t^{p-1}). \qquad \blacksquare$$

Theorem 4.1 *Let $v \in CMC^{N,m}$. Then the constrained egalitarian allocation $(d_{ij})_{i \in N, j \in M_i^+}$ belongs to the precore $\mathcal{PC}(v)$ of v.*

Proof. Let P be the number of steps in Dutta-Ray algorithm, $t^1, t^2, ..., t^P$ be the corresponding sequence of multi-choice coalitions in $\mathcal{M}_+^N$, and $\alpha_1, \alpha_2, ..., \alpha_P$ be the sequence of average values of these coalitions. Note that each $s \in \mathcal{M}_+^N$ can be expressed as

$$s = (s \wedge t^1) + (s \wedge t^2 - s \wedge t^1) + ... + (s \wedge t^P - s \wedge t^{P-1}),$$

where some of the terms could be zero. Then, by definition of $D(s)$ and α_p, $p \in \{1, ..., P\}$, $D(s)$ can be rewritten as follows:

$$\begin{aligned}
&D(s) = \sum_{i \in N} \sum_{j=1}^{s_i} d_{ij} \\
&= \|s \wedge t^1\|_1 \alpha_1 + \|s \wedge t^2 - s \wedge t^1\|_1 \alpha_2 + ... + \|s \wedge t^P - s \wedge t^{P-1}\|_1 \alpha_P \\
&= \|s \wedge t^1\|_1 \frac{v(t^1)}{\|t^1\|_1} + \|s \wedge t^2 - s \wedge t^1\|_1 \frac{v(t^2) - v(t^1)}{\|t^2 - t^1\|_1} \\
&+ ... + \|s \wedge t^P - s \wedge t^{P-1}\| \frac{v(t^P) - v(t^{P-1})}{\|t^P - t^{P-1}\|_1}.
\end{aligned}$$

Now, in view of Remark 4.1, we obtain

$$\begin{aligned}
&D(s) \geq \|s \wedge t^1\|_1 \frac{v(t^1)}{\|s \wedge t^1\|_1} + \|s \wedge t^2 - s \wedge t^1\|_1 \frac{v((s \wedge t^2) - (s \wedge t^1) + t^1) - v(t^1)}{\|s \wedge t^2 - s \wedge t^1\|_1} \\
&+ ... + \|s \wedge t^P - s \wedge t^{P-1}\|_1 \frac{v((s \wedge t^P) - (s \wedge t^{P-1}) - t^{P-1}) - v(t^{P-1})}{\|s \wedge t^P - s \wedge t^{P-1}\|_1} \\
&\geq \|s \wedge t^1\|_1 \frac{v(s \wedge t^1)}{\|s \wedge t^1\|_1} + \|s \wedge t^2 - s \wedge t^1\|_1 \frac{v(s \wedge t^2) - v(s \wedge t^1)}{\|s \wedge t^2 - s \wedge t^1\|} \\
&+ ... + \|s \wedge t^P - s \wedge t^{P-1}\|_1 \frac{v(s \wedge t^P) - v(s \wedge t^{P-1})}{\|s \wedge t^P - s \wedge t^{P-1}\|_1} \\
&= v(s \wedge t^1) + (v(s \wedge t^2) - v(s \wedge t^1)) + ... + (v(s \wedge t^P) - v(s \wedge t^{P-1})) \\
&= v(s \wedge t^P) = v(s),
\end{aligned}$$

where the last inequality follows from Lemma 4.2. Hence, $D(s) \geq v(s)$ for each $s \in \mathcal{M}_+^N$. Finally, $D(m) = v(m)$ follows from the constructive definition of d, too. ∎

The constrained egalitarian solution of a multi-choice game does not necessarily belong to the imputation set of the game, as we illustrate in the next example.

Example 4.2 Consider the convex multi-choice game $\langle N, m, v\rangle$, with $N = \{1,2\}$, $m = (3,2)$, and $v(0,0) = 0$, $v(1,0) = v(0,1) = 1$, $v(2,0) = v(1,1) = v(0,2) = 2$, $v(2,1) = v(1,2) = 3$, $v(3,0) = v(2,2) = 5$, $v(3,1) = 6$, $v(3,2) = 12$. The constrained egalitarian allocation is $d(v) = (2.4, 2.4, 2.4, 2.4, 2.4)$. Note that $d_{13} = 2.4 < v(3e^1) - v(2e^1) = 5 - 2 = 3$. Hence, $d(v) \notin I(v)$.

Theorem 4.2 *Let $v \in CMC^{N,m}$. The constrained egalitarian solution $d(v)$ of v Lorenz dominates each other precore element of v.*

Proof. Let $x \in \mathcal{PC}(v)$. We prove that $x \succ_L d$ implies $x = d$. Suppose that $0 = t^0, t^1, ..., t^P = m$ is the sequence of the Dutta-Ray multi-choice coalitions in $\mathcal{M}^N$ for d (see Remark 4.1). We use backward induction to prove two assertions:

(i) (Induction basis) $d_{ij} = x_{ij}$ for each $(i,j) \in (t^{P-1}, t^P]$, i.e. for all j s.t. $t_i^{P-1} < j \le t_i^P$, $i \in N$.

(ii) (Induction step) For each $k \in \{P-1, ..., 1\}$ it holds true that if $d_{ij} = x_{ij}$ for each $(i,j) \in (t^k, t^P]$, then $d_{ij} = x_{ij}$ for each $(i,j) \in (t^{k-1}, t^P]$.

By Proposition 4.2, the smallest elements of $d = (d_{ij})_{i \in N, j \in M_i^+}$ correspond precisely to elements $(i,j) \in (t^{P-1}, t^P]$, and there

$$d_{ij} = \frac{v(t^P) - v(t^{P-1})}{\|t^P - t^{P-1}\|_1} = \alpha_P.$$

Since $x \succ_L d$, it follows that $x_{ij} \ge d_{ij}$ for all $(i,j) \in (t^{P-1}, t^P]$, implying that

$$\begin{aligned} x((t^{P-1}, t^P]) &= \sum_{(i,j)\in(t^{P-1},t^P]} x_{ij} \ge \sum_{(i,j)\in(t^{P-1},t^P]} d_{ij} \\ &= d((t^{P-1}, t^P]) = \alpha_P \|t^P - t^{P-1}\|_1. \end{aligned}$$

Suppose that (i) does not hold. Then, we obtain

$$x((t^{P-1}, t^P]) > d(t^{P-1}, t^P] = v(t^P) - v(t^{P-1}).$$

But, since $x \in \mathcal{PC}(v)$, we also have

$$\begin{aligned} & x((t^{P-1}, t^P]) = x((0, t^P]) - x((0, t^{P-1}]) = v(t^P) - x((0, t^{P-1}]) \\ & \le v(t^P) - v(t^{P-1}) = \alpha_P \|t^P - t^{P-1}\|_1, \end{aligned}$$

where the second equality follows from the efficiency condition for precore elements and the inequality from the stability conditions for precore elements. So, we conclude that (i) holds true.

Now, we prove (ii). Suppose $d = x$ on $(t^k, t^P]$. Then, by Proposition 4.2, the worst $\|t^P - t^k\|_1$ elements of d and x are in $(t^k, t^P]$ and "coincide". Since $x \succ_L d$ we have:

(1) $x_{ij} \geq \alpha_k = d_{ij}$ for all $(i,j) \in (t^{k-1}, t^k]$, $k \in \{1, ..., P\}$,

and

(2) $x((0,t^k]) = \sum_{(i,j)\in(0,t^k]} x_{ij} = v(t^k)$ because $v(t^P) - v(t^k) = d((t^k, t^P]) = x((t^k, t^P])$ and $v(t^P) = x((0, t^P])$.

Then, $x \in \mathcal{PC}(v)$ implies

(3) $x((t^{k-1}, t^k]) = x((0,t^k]) - x((0,t^{k-1}]) \leq v(t^k) - v(t^{k-1}) = \alpha_k \|t^k - t^{k-1}\|_1$,

where the inequality follows from (2) and the stability conditions. Then, from (1) and (3) it follows that $x_{ij} = d_{ij}$ for $(i,j) \in (t^{k-1}, t^k]$, and so, $x = d$ on $(t^{k-1}, t^P]$. ∎

Theorem 4.3 *Let $v \in CMC^{N,m}$. Then $d(v) \in EDC(v)$.*

Proof. According to Theorems 3.1 and 4.1, we have $d(v) \in \mathcal{PC}(v) \subset EDC(v)$. ∎

Remark 4.2 In Peters and Zank (2005), a Shapley-type value, called the egalitarian solution for multi-choice games, was introduced and axiomatically characterized by the properties of efficiency, zero contribution, additivity, anonymity, and level-symmetry. However, this solution concept has no connection with the average worth of a multi-choice coalition and with the Lorenz criterion, and makes incomplete use of information regarding the characteristic function. This entitles us to claim that on the class of convex multi-choice games the egalitarian solution neither coincides with the constrained egalitarian solution nor belongs to the equal division core.

Given Theorem 4.3, it is not difficult to provide an axiomatic characterization of the constrained egalitarian solution on the class of convex multi-choice games in line with Theorem 3.3 of Klijn et al. (2000), using the properties of efficiency, equal division stability and max-consistency. We start with introducing the multi-choice counterparts of these properties.

Given a single-valued solution $\Psi : CMC^{N,m} \to \mathbb{R}^{\sum_{i \in N} m_i}$, we denote by s^m the multi-choice coalition such that for each $i \in N$, for all $k \in \{1, ..., s_i^m\}$, $\Psi_{ik}(v) = \max_{(i,j) \in M^+} \Psi_{ij}(v)$. We say that Ψ satisfies

- *Efficiency*, if for all $v \in CMC^{N,m}$: $\sum_{i \in N} \sum_{j=1}^{m_i} \Psi_{ij}(v) = v(m)$;
- *Equal division stability*, if for all $v \in CMC^{N,m}$: $\Psi(v) \in EDC(v)$;
- *Max-consistency*, if for all $v \in CMC^{N,m}$ and all $(i,j) \in M^+$, $\Psi_{ij}(v) = \Psi_{ij}(v^{-s^m})$, where v^{-s^m} is the multi-choice game defined by $v^{-s^m}(t) = v(t + s^m) - v(s^m)$ for all $t \in \mathcal{M}^N_{m-s^m}$.

Theorem 4.4 *There is a unique solution on $CMC^{N,m}$ satisfying the properties efficiency, equal division stability and max consistency, and it is the constrained egalitarian solution.*

5 On the equal split-off set for multi-choice games

In Branzei, Dimitrov and Tijs (2006) the equal split-off set ($ESOS$) was introduced for arbitrary cooperative games $v \in G^N$ as a new set valued solution concept based on egalitarian considerations and inspired by the Dutta-Ray algorithm (cf. Dutta and Ray (1989)). It was proved that for superadditive games $v \in G^N$ the equal split-off set is a refinement of the equal division core, and for each game $v \in CG^N$ the equal split-off set $ESOS(v)$ consists of a single allocation which is the Dutta-Ray constrained egalitarian solution $E(v)$ of that game. The multi-choice extension of the $ESOS$ can be obtained by adjusting, in the same spirit as in Branzei, Dimitrov and Tijs (2006), for arbitrary multi-choice games the Dutta-Ray algorithm for convex multi-choice games. Specifically, in each step of the new procedure, one of the multi-choice coalitions with the highest (per one-unit level increase) average value is chosen and the corresponding levels divide equally the value of that coalition. We note that such multi-choice coalitions need not be the largest coalitions with the highest average value. Moreover, in a straightforward way, but technically cumbersome, we can also extend the properties of $ESOS$ for traditional cooperative games to its multi-choice counterpart. In particular, for each convex multi-choice game $v \in MC^{N,m}$, the equal split-off set $ESOS(v)$ consists of a single element which is the constrained egalitarian solution $d(v)$ of v.

References

[1] Arin, J. and E. Inarra (2001): Egalitarian solutions in the core, International Journal of Game Theory 30, 187-193.

[2] Arin, J. and E. Inarra (2002): Egalitarian sets for TU-games, International Game Theory Review 4, 183-199.

[3] Atkinson, A.B. (1970): On the Measurement of Inequality, Journal of Economic Theory 2, 244-263.

[4] Bhattacharya, A. (2004): On the equal division core, Social Choice and Welfare 22, 391-399.

[5] Bondareva, O.N. (1963): Certain applications of the methods of linear programming to the theory of cooperative games, Problemy Kibernetiki 10, 119-139 (in Russian).

[6] Branzei, R., D. Dimitrov, and S. Tijs (2005): Models in Cooperative Game Theory: Crisp, Fuzzy, and Multi-Choice Games, Lecture Notes in Economics and Mathematical Systems, vol. 556, Springer.

[7] Branzei, R., D. Dimitrov, and S. Tijs (2006): The equal split-off set for cooperative games, Game Theory and Mathematical Economics, Banach Center Publications 71, 39-46.

[8] Branzei, R., S. Tijs, and J. Zarzuelo (2007): Convex multi-choice cooperative games and their monotonic allocation schemes, CentER DP 2007, Tilburg University, The Netherlands.

[9] Dasgupta, P., A.K. Sen, and D. Starret (1973): Notes on the Measurement of Inequality, Journal of Economic Theory 6, 180-187.

[10] Dutta, B. and D. Ray (1989): A concept of egalitarianism under participation constraints, Econometrica 57, 615-635.

[11] Dutta, B. (1990): The egalitarian solution and reduced game properties in convex games, International Journal of Game Theory 19, 153-169.

[12] Dutta, B. and D. Ray (1991): Constrained egalitarian allocations, Games and Economic Behavior 3, 403-422.

[13] Fields, G. and J. Fei (1978): On Inequality Comparisons, Econometrica 46, 303-316.

[14] Gillies, D.B. (1953): Some theorems on n-person games, Ph. D. Thesis, Princeton University Press, Princeton, New Jersey, 1953.

[15] Grabisch, M. and L. Xie (2007): A new investigation about the core and Weber set of multichoice games, Mathematics of Operations Research, forthcoming.

[16] Hokari, T. (2000): Population monotonic solutions for convex games, International Journal of Game Theory 29, 327-338.

[17] Hougaard, J., B. Peleg, and L. Thorlund-Petersen (2001): On the set of Lorenz-maximal imputations in the core of a balanced game, International Journal of Game Theory 30, 147-165.

[18] Hsiao, C.-R. and TES Raghavan (1993a): Monotonicity and dummy free property for multi-choice cooperative games, International Journal of Game Theory 21, 301-312.

[19] Hsiao, C.-R. and TES Raghavan (1993b): Shapley value for multi-choice cooperative games (I), Games and Economic Behavior 5, 240-256.

[20] Klijn, F., M. Slikker, and J. Zarzuelo (2000): The egalitarian solution for convex games: some characterizations, Mathematical Social Sciences 40, 111-121.

[21] Nouweland van den, A., J. Potters, S. Tijs, and J. Zarzuelo (1995): Cores and related solution concepts for multi-choice games, Mathematical Methods of Operations Research 41, 289-311.

[22] Peters, H. and H. Zank (2005): The egalitarian solution for multi-choice games, Annals of Operations Research 137, 399-409.

[23] Selten, R. (1972): Equal share analysis of characteristic function experiments, in: Sauermann, H. (Ed.), Contributions to Experimentation in Economics, Vol. 3, Mohr Verlag, pp.130-165.

[24] Sen, A.K. (1973): On Economic Inequality, Oxford: Clarenden Press.

[25] Sen, A. K. (1997): On Economic Inequality, Oxford: Oxford University Press.

[26] Shapley, L.S. (1967): On balanced sets and cores, Naval Logistics Quarterly 14, 453-460.

[27] Shapley, L.S. (1971): Cores of convex games, International Journal of Game Theory 1, 11-26.

[28] Sharkey, W.W. (1982): Cooperative games with large cores, International Journal of Game Theory 11, 175-182.

[29] Thomson, W. (1995): Population-monotonic allocation rules. In: Barnett,W.A., H. Moulin, M. Salles, and N. J. Schofield (Eds.), Social Choice, Welfare and Ethics, Cambridge University Press, Cambridge, p. 79-124.